STUDY OF STATISTICAL CONVERGENCE OF TRIPLE SEQUENCES IN A TOPOLOGICAL SPACE

Estudio de la convergencia estadística de secuencias triples en un espacio topológico

BIROJIT DAS[1], APURBA DEBNATH[2], CARLOS GRANADOS[3], ABDULLAH AYDIN[4]

Recibido:20 de mayo de 2024. Aceptado:31 de enero de 2025

DOI: https://doi.org/10.21017/rimci.1135

Abstract

In this paper, we introduce statistical convergence of triple sequences which are defined in a topological space. Various results on statistically convergent triple sequences are produced by using the notion of triple natural density operator. Moreover, we initiate the concept of s*-convergence of triple sequence and establish the interrelationship among s*-convergence and s-converegnce. We see that the first one implies the second one but it's not vice-versa. But if we restrict the triple sequences to hold the property of first countability, we verify that these notions becomes equivalent. Finally, we prove that the family of all statistically convergent triple sequences under some conditions generates a topological structure within the topological space where they have been defined.

Keywords: statistical convergence; triple sequence; topological space.

Resumen

En este artículo, introducimos la convergencia estadística de sucesiones triples definidas en un espacio topológico. Se obtienen diversos resultados sobre sucesiones triples estadísticamente convergentes utilizando la noción del operador de densidad natural triple. Además, iniciamos el concepto de convergencia s* de sucesiones triples y establecemos la interrelación entre la convergencia s* y la convergencia s. Observamos que la primera implica la segunda, pero no ocurre lo contrario. Sin embargo, si restringimos las sucesiones triples a cumplir con la propiedad de primer numerabilidad, verificamos que estas nociones se vuelven equivalentes. Finalmente, demostramos que la familia de todas las sucesiones triples estadísticamente convergentes, bajo ciertas condiciones, genera una estructura topológica dentro del espacio topológico en el cual han sido definidas.

Palabras clave: convergencia estadística, sucesión triple, espacio topológico.

I. Introducción

The initial study of statistical convergence of real / complex sequences started back in the mid of twentieth century. At that time this conception was extensively examined by Fast[1], Schohenberg[2], Buck[3] independently. In the next three decades researchers had shown little interest in exploring this notion. But once the articles on statistical convergence by Fridy[4] and Salat[5], came into literature, it got the real momentum. After that many researchers introduced this convergence concept of sequences in various environments in different aspects. It has been explored in different types of

1 Department of Mathematics, National Institute of Technology Warangal, India. ORCID: https://orcid.org/0000-0001-7614-1051

2 Department of Mathematics, National Institute of Technology Agartala, India. ORCID: https://orcid.org/0000-0002-0541-8241

3 Escuela Ciencias de la Educación, Universidad Nacional Abierta y a Distancia, Barranquilla Colombia. ORCID: https://orcid.org/0009-0007-1305-7822 Correo electrónico: carlosgranadosortiz@outlook.es

4 Department of Mathematics, Mus Alparslan University & Middle East Technical University, Turkey. ORCID: https://orcid.org/0000-0002-0769-5752

spaces like locally solid Riesz space by Mohiuddine & Alghamdi[6], Albayrak & Pehlivan[7], topological group by Cakalli[8], Maio & Kocinac [9], intutionistic fuzzy normed space by Karakus, Demirci, & Duman[10], Kumar & Mursaleen[11], Mohiuddine & Savas[12], locally convex space by Maddox[13] and so on. The notion of statistical convergence in double sequence has been initiated by Tripathy[14], Mursaleen & Edely[15] and Moricz[16] independently around the same time. Several mathematicians studied statistical convergence of real or complex double sequences via summability since starting of twenty first century, for instance Belen, Mursaleen, & Yildirim[17], Demirci & Karakus[18][19], Edely & Mursaleen[20]. In the same context, Mursaleen, Cakan, & Mohiuddine[21] established more results on statistical convergence of double sequences by introducing generalized statistical convergence and statistical core. In the year 2007, Karkaus & Demirci[22] extended the study of statistical convergence in probabilistic normed spaces and after those different forms of statistical convergence of double sequences like lacunary statistical convergence introduced by Mohiuddine & Savas[23], generalized statistical convergence Savas & Mohiuddine[24], λ-statistical convergence Savas & Mohiuddine[24] and many others are studied in the same environment.

In the uncertain environment, Tripathy & Nath[25] introduced the concept of statistical convergence by considering sequences of complex uncertain variables. Das, Tripathy, Debnath, & Bhattacharya[26][27], Das, Tripathy, & Debnath[28], Das, Debnath, & Tripathy[29], Das & Tripathy[30] extended this study by considering complex uncertain double and triple sequences in all five aspects of uncertainty. The main aim of this research work is to introduce the same notion of statistical convergence in a topological space by considering triple sequences. We establish various properties of statistically convergent triple sequence defined in a given topological space up to some extent. We also initiate the concept of s^*-convergent triple sequence in the same structure and show that s^*-convergence of a triple sequence implies its statistical convergence therein. At the end we show that the collection of all statistically convergent triple sequences under some given conditions generates a topological structure on a non-empty subset of the topological space in which the triple sequences are defined.

The main objective of this study is to generalize the concept of statistical convergence for triple sequences within a topological space and explore their key properties. The specific aims include:

(i) To define and analyze statistical convergence for triple sequences in a topological context.

(ii) To introduce and study the notion of s^*-convergence and its relation to statistical convergence.

(iii) To demonstrate that the family of statistically convergent triple sequences can generate a topological.

Therefore, this manuscript contributes significantly to the field of mathematical analysis by extending the concept of statistical convergence to triple sequences in topological spaces—a direction scarcely explored in current literature. The importance of this contribution lies not only in the novelty of the triple sequence approach but also in its theoretical robustness, which paves the way for further generalizations in functional analysis, topology, and uncertainty modeling. Technologically, the proposed framework could support computational models that handle multi-indexed data in complex systems[31]; Das et al.[32]. Methodologically, it provides an expanded toolbox for analyzing convergence in abstract spaces, which is crucial for the development of advanced theories in summability, fuzzy logic, and neutrosophic analysis[15];[33];[5]. These theoretical advancements not only enrich the mathematical discourse but also offer applied benefits in areas like signal processing, stochastic modeling, and artificial intelligence where multidimensional uncertainty and convergence behavior play essential roles[23];[34].

The methodology is based on theoretical mathematical analysis, utilizing topological properties, density operators, and convergence principles. We follow a deductive and axiomatic framework to establish definitions, propositions, and theorems relevant to the convergence behavior of triple sequences.

Thus, we now procure some fundamental concepts and results on triple sequences; those will be used throughout the paper.

II. Preliminaries

We know that the study of statistical convergence of sequences based on the concept of natural density. In the following we present the very basic definition of convergence of triple sequence and statistical convergence of the same via triple natural density.

a. Definition[35]

A real triple sequence $\{x_{nkl}\}$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$|x_{nkl} - L| < \varepsilon, \text{ whenever } n \geq N, k \geq N, l \geq N$$

b. Definition[35]

A subset K of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have a natural density (sometimes we call asymptotic density) $\delta_3(K)$ if

$$\delta_3(K) = \lim_{p,q,r\to\infty} \frac{|K(p,q,r)|}{pqr}, \text{ exists}$$

where, the vertical bars denote the numbers of (l, m, n) in K such that $l \leq p, m \leq q$ and $n \leq r$.

c. Definition[35]

A real triple sequence $\{x_{nkl}\}$ is said to be statistically convergent to the number L if for each $\varepsilon > 0$

$$\delta_3(\{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: |x_{nkl} - L| \geq \varepsilon\}) = 0.$$

III. Main results

In this section, at first, we introduce statistical convergence of triple sequences which are defined in a topological space. Various results on statistically convergent triple sequences are produced by using the notion of triple natural density operator. Moreover, we initiate the concept of s^*-convergence of triple sequence and establish the interrelationship among s^*-convergence and s-converegnce. We see that the first one implies the second one but it's not vice-versa. But if we restrict the triple sequences to hold the property of first countability, we verify that these notions become equivalent. Finally, we prove that the family of all statistically convergent triple sequences under some conditions generates a topological structure within the topological space where they have been defined.

a. Definition

A set A in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is called the statiscally dense if it has unit triple asymptotic density.

It is obvious that

$d(\mathbb{N} \times \mathbb{N} \times \mathbb{N}) = 1$ and so $d(\mathbb{N} \times \mathbb{N} \times \mathbb{N} - A) = 1 - d(A)$,
for every $A \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

In the following, we consider an example, where we observe a triple sequence having zero triple density.

b. Example

Let us consider a triple sequence $\{x_{lmn}\}$ defined as following

$$x_{lmn} = \begin{cases} 1, & if\ l < m < n\ are\ prime \\ 0, & otherwise \end{cases}$$

We take a δ-neighborhood of 0 and see that

$$d(\{(l, m, n) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: x_{lmn} \notin A\})$$

$$= \lim_{l,m,n\to\infty} \frac{|\{(l, m, n) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: x_{lmn} \notin A)\}|}{lmn} = 0$$

Therefore, the above set has asymptotic triple density 0.

c. Definition

A triple sequence $\{x_{lmn}\}$ in a topological space $(X, \mathfrak{I})$ is said to be a statistically convergent triple sequence to a point $x \in X$ (in short, s-converence) if for any nbd U of x, the set $\{(l, m, n) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: x_{lmn} \notin U\}$ has zero asymptotic triple density.

If a triple sequence $\{x_{lmn}\}$ s-converges to x, then we write

$$x = s\text{-}\lim_{l,m,n\to\infty} x_{lmn} \text{ or } x_{lmn} \to_s x.$$

d. Remark

If we take the usual topology on real line, then from the above Definition 3.3, we can deduce that the set $A = \{(l, m, n) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\}$ with unit triple natural density such that the triple sequence $\{x_{lmn}\}$ converges to x, in Pringsheim's sense, i.e., for any nbd U of X, there exists a natural number m_0 such that for all $l, m, n \geq m_0$ and $(l, m, n \in A)$, we have $x_{lmn} \in U$.

e. Definition

Let $\{x_{l_i m_j n_k}\}$ be a subsequence of the triple sequence $\{x_{lmn}\}$ and $K = \{(l_i, m_j, n_k: i, j, k \in \mathbb{N})\}$.

If $d(k) = 0$, we say that the subsequence has zero triple density or a thin triple sequence.

f. Definition

A subsequence $\{x_{l_i m_j n_k}\}$ of a triple sequence $\{x_{lmn}\}$ is said to be statistically dense in $\{x_{lmn}\}$ if the set $\{(l_i, m_j, n_k: i, j, k \in \mathbb{N})\}$ is statistically dense in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$:.

g. Definition

Let $\{x_{lmn}\}$ be a triple sequence in a topological space $(X, \mathfrak{I})$. The triple sequence is said to be s^*-convergent to a point x in X if there exists a set A having unit triple natural density such that

$$\lim_{\substack{l,m,n\to\infty \\ (l,m,n\in A)}} x_{lmn} = x$$

and we write $s^*\text{-}\lim x_{lmn} = x$.

In the following, we present an important result which gives the relation between convergence and s-convergence of triple sequence.

h. Theorem

If a triple sequence $\{x_{lmn}\}$ in a topological space $(X, \mathfrak{I})$ converges to a point $x \in X$, then the triple sequence statistically converges to the same point.

Proof: Let the triple sequence is convergent to the limit x in a topological space $(X, \mathfrak{I})$. Thus for every neighborhood U of x, there exists a natural number n_0 such that $x_{lmn} \in U, \forall\, l, m, n \geq n_0$.

Then,

$$\{(l, m, n): x_{lmn} \notin U\} \subset (\mathbb{N} \times \mathbb{N} \times \mathbb{N}) - A.$$

Now

$$d(\mathbb{N} \times \mathbb{N} \times \mathbb{N} - A) = 1 - d(A) = 1 - 1 = 0$$

Thus,

$$d(\{(l, m, n): x_{lmn} \notin U\}) = 0$$

Hence triple sequence $\{x_{lmn}\}$ statistically converges to x.

i. Lemma

The converse of the above Theorem 3.8 is not true in general. This claim is verified in the following example.

j. Example

Let us consider usual topological space $\mathbb{R}$ and within it we define a triple sequence $\{x_{lmn}\}$ as follows:

$$x_{lmn} = \begin{cases} 1, & if\ l, m\ both\ prime \\ 2, & if\ m, n\ both\ prime \\ 3, & if\ l, n\ both\ prime \\ 4, & if\ l, m, n\ all\ are\ prime \\ 0, & otherwise \end{cases}$$

We know that set of prime natural number has zero triple natural density.

Therefore, the triple sequence s-converges to $x = 0$.

But for obvious reason it doesn't converges to any limit at all.

k. Theorem

If a triple sequence is statistically convergent to a certain limit in a given topological space, then each non-thin subsequence of it is also s -convergent by preserving the limit therein.

Proof: Let $\{x_{lmn}\}$ be a triple sequence in a topological space $(X, \mathfrak{I})$ which s-converges to a point $x \in X$ and $\{x_{l_i m_j n_k}\}$ be a non thin sequence of it.

Let U be the neighborhood of the point x. Then, $d(\{(l, m, n): x_{lmn} \notin U\}) = 0$

We label the sub sequential element $x_{l_i m_j n_k}$ by y_{ijk} and take g, be a one-one mapping in such a way that

$$g(y_{ijk}) = x_{l_i m_j n_k}.$$

We are to prove that $y_{ijk} \overset{s}{\rightarrow} x$

For that we consider a subsequence $\{y_{i_p j_q k_r}\}$ in such a way that $z_{pqr} = y_{i_p j_q k_r} \in U$ and an injective function f such that $f(z_{pqr}) = y_{i_p j_q k_r}$

$$\Rightarrow (gof)\,(z_{pqr}) = x_{l_{i_p} m_{j_q} n_{k_r}}.$$

Let $d\left(\left\{(l_i, m_j, n_k):\ y_{ijk} \in g^{-1}\left(x_{l_i m_j n_k}\right)\right\}\right) = t$

Since, $x_{lmn} \overset{s}{\rightarrow} x$, so

$$d\left(\left\{(l_i, m_j, n_k):\ g(y_{ijk}) = x_{l_i m_j n_k} \in U\right\}\right) = t$$

i.e., $d\left(\left\{\left(l_{i_p}, m_{j_q}, n_{k_r}\right):\ z_{pqr} \in (gof)^{-1}(x_{l_{i_p} m_{j_q} n_{k_r}})\right\}\right) = t$

Thus, $\lim_{u,v,w\to\infty} \frac{\left|\left(l_{i_p}, m_{j_q}, n_{k_r}\right): z_{pqr} \in (gof)^{-1}\left(x_{l_{i_p} m_{j_q} n_{k_r}}\right),\ l_{i_p} \le u,\ m_{j_q} \le v, n_{k_r} \le w\right|}{u.v.w} = t$

Now, if $\left|\left\{(l_i, m_j, n_k):\ y_{ijk} \in g^{-1}\left(x_{l_{i_p} m_{j_q} n_{k_r}}\right), l_{i_p} \le u,\ m_{j_q} \le v,\ n_{k_r} \le w\right\}\right| = ijk$

Then we have,

$$\left\{\left(l_{i_p}, m_{j_q}, n_{k_r}\right):\ z_{pqr} \in (gof)^{-1}(x_{l_{i_p} m_{j_q} n_{k_r}})\right\} = \left\{(i_p, j_q, k_r): i_p \le i, j_q \le j, k_r \le k,\ z_{pqr} \in f^{-1}\left(y_{i_p j_q k_r}\right)\right\}$$

Then we can say that

$$t = \lim_{l,m,n\to\infty} \frac{\frac{\left|\left(l_{i_p}, m_{j_q}, n_{k_r}\right): z_{pqr} \in (gof)^{-1}\left(x_{l_{i_p} m_{j_q} n_{k_r}}\right), \ l_{i_p} \le l, \ m_{j_q} \le m, n_{k_r} \le n\right|.}{\left|\left\{\left(l_i, m_j, n_k\right): \ y_{ijk} \in g^{-1}\left(x_{l_{i_p} m_{j_q} n_{k_r}}\right)\right\}, l_i \le l, \ m_j \le m \ n_k \le n\right|}}{l.m.n.i.j.k}$$

$$= \lim_{i,j,k\to\infty} \frac{\left|\left(i_p, j_q, k_r\right): z_{pqr} \in f^{-1}\left(y_{l_{i_p} m_{j_q} n_{k_r}}\right), \ i_p \le i, \ j_q \le j, k_r \le k\right|}{i.j.k}.$$

$$\lim_{l,m,n\to\infty} \frac{\left|\left\{\left(l_i, m_j, n_k\right): \ y_{ijk} \in g^{-1}\left(x_{l_{i_p} m_{j_q} n_{k_r}}\right)\right\}, l_i \le l, \ m_j \le m \ n_k \le n\right|}{l, m, n}$$

$$= \lim_{i,j,k\to\infty} \frac{\left|\left(i_p, j_q, k_r\right): z_{pqr} \in f^{-1}\left(y_{l_{i_p} m_{j_q} n_{k_r}}\right), \ i_p \le i, \ j_q \le j, k_r \le k\right|}{i.j.k} . t$$

Therefore , $\lim_{i,j,k\to\infty} \frac{\left|\left(i_p, j_q, k_r\right): z_{pqr} \in f^{-1}\left(y_{l_{i_p} m_{j_q} n_{k_r}}\right), \ i_p \le i, \ j_q \le j, k_r \le k\right|}{i.j.k} = 1$

and so $d\left(\left\{(i, j, k): y_{ijk} \in U\right\}\right) = 1$

i.e., $d\left(\left\{(i, j, k): y_{ijk} \notin U\right\}\right) = 0$

Hence the subsequence $\{y_{ijk}\}$ s-converges to x.

l. Remark

If we drop the condition non-thinness from the subsequence of a statistically convergent triple sequence, then it may not be s-convergent therein.

This claim is justified in the following example.

m. Example

We consider a triple sequence $\{x_{lmn}\}$ defined in the usual real line topology $\mathbb{R}$ as follows:

$$x_{lmn} = \begin{cases} mn, & if \ m \ and \ n \ are \ square \ number \\ ln, & if \ l \ and \ n \ are \ square \ number \\ lm, & if \ l \ and \ m \ are \ square \ number \\ \frac{1}{lmn}, & otherwise \end{cases}$$

It is obvious that the triple sequence converges statistically to $x = 0 \in \mathbb{R}$

We now consider a subsequence $\{x_{l_i m_j n_k}\}$ of $\{x_{lmn}\}$ which is defined by $x_{l_i m_j n_k} = ijk$

It means that i, j, k are square numbers.

Thus the subsequence $\{x_{l_i m_j n_k}\}$ is a thin triple sequence. $[Since \ d(\{(i^2, j^2, k^2)\}) = 0]$

Hence, the triple sequence is not a s - convergent triple sequence.

n. Theorem

A triple sequence is s-convergent iff all of its statistically dense subsequence is also s - convergent.

Proof: The necessary part is proved already in the above theorem.

The converse part can be proved very easily from the fact that every triple sequence is statistically dense in itself.

o. Theorem

s^*-convergence of a triple sequence in a given topological space implies its s-convergence.

Proof: Let $\{x_{l,mn}\}$ be s^*-convergent triple sequence in a topological space $(X,\mathfrak{I})$ to $x \in X$.

Then, for any neighborhood U of x, there exists $A \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with unit triple asymptotic density and $l_0, m_0, n_0 \in \mathbb{N}$, such that

$$x_{lmn} \in U, \forall\, l \geq l_0, m \geq m_0, n \geq n_0.$$

Then, we have $\{(l,m,n): x_{lmn} \notin U\}$

$$\subset \{(l_0, m_0, n_0)\} \cup (\mathbb{N} \times \mathbb{N} \times \mathbb{N} - \mathrm{A})$$

Since, $d\big(\{(l_0, m_0, n_0)\} \cup (\mathbb{N} \times \mathbb{N} \times \mathbb{N} - \mathrm{A})\big) = 0$, so, $d(\{(l,m,n): x_{lmn} \notin U\}) = 0$.

p. Remark

Converse of the above Theorem 3.15 is not true. This is illustrated in the example given below.

q. Example

Instead of taking usual topology on $\mathbb{R}$ taken in the example considered above, here we define a triple sequence $\{x_{lmn}\}$ in a topological space $X = C \cup \{0,1\}$, where C is a uncountable set, on which the co-finite topology is structured.

Let, the triple sequence be defined on a set $A \subset X$ by,

$$x_{lmn} = \begin{cases} 1, & if\ l,m,n\ are\ infinite \\ 0, & otherwise \end{cases} \quad \text{------------} \quad 1$$

For obvious reason $d(A) = 1$, $[Since\ d(\mathbb{N}) = 1]$

From equation 1, it can be seen that the triple sequence doesn't converge to any limit and hence it is not s^*-convergenrt.

r. Remark

Although in the Remark 3.16, we claimed that s-convergence doesn't imply s^*-convergent in general. But if we add an additional condition of countability to the topological space then the reverse implication of the Theorem 3.15 becomes true

s. Theorem

Let $\{x_{lmn}\}$ be a statistically convergent triple sequence in a first countable topological space, then it is a s^*-convergent triple sequence with the limit preservation property.

Proof: Let $U_1 \supset U_2 \supset \cdots \ldots \ldots$ be a countable base at the s-converging point $x \in X$ of the triple sequence $\{x_{lmn}\}$.

For any $a \in \mathbb{N}$, we assign $A_a = \{(l,m,n): x_{lmn} \in \mathrm{U_a}\}$.

Thus, we have $A_1 \supset A_2 \supset \cdots \ldots$ and each of the set has triple asymptotic density unity.

We suppose $(i_1, j_1, k_1) \in A_1$. Then, there exists $(i_2, j_2, k_2) \in A_2$ with $i_2 \geq i_1, j_2 \geq j_1, k_2 \geq k_1$ such that for all $(l,m,n) \geq (i_2, j_2, k_2)$, we have

$$\frac{|A_2(l,m,n)|}{lmn} = \frac{|\{(p,q,r) \in A_2 : p \leq l, q \leq m, r \leq n\}|}{lmn} > \frac{1}{2}$$

$$[Since\ d(A_2) = 1]$$

In the similar manner, we get $(i_a, j_a, k_a) \in A_a$ with $(i_1, j_1, k_1) \leq (i_2, j_2, k_2) \leq \cdots \ldots \ldots \leq (i_a, j_a, k_a)$ such that for all $l \geq i_a, m \geq j_a\ and\ n \geq k_a$,

$$\frac{|A_a(l,m,n)|}{lmn} = \frac{|\{(p,q,r) \in A_a : p \leq l, q \leq m, r \leq n\}|}{lmn} > 1 - \frac{1}{a}.$$

We consider the set $A \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ in such a way that if $(i,j,k) \le (i_1, j_1, k_1)$, we have $(i,j,k) \in A$ and whenever $(i,j,k) \le (i_1, j_1, k_1)$, $(i,j,k) \in A$ if $j = 1$ and whenever $(i_a, j_a, k_a) \le (i,j,k) \le (i_{a+1}, j_{a+1,} k_{a+1})$, $(i,j,k) \in A$ iff $(i,j,k) \in A_a$

$$\text{Let } A = \{(l_1, m_1, n_1): l_1 < l_2 < \cdots, m_1 < m_2 < \cdots, n_1 < n_2 < \cdots\}.$$

In this scenario, if $(i_a, j_a, k_a) < (l,m,n) \le (i_{a+1}, j_{a+1,} k_{a+1})$, then we can write

$$\frac{|A(l,m,n)|}{lmn} \ge \frac{|A_a(l,m,n)|}{lmn} > 1 - \frac{1}{a}$$

and thus triple natural density of A is 1.

We now show that $\lim_{n,m,l \to \infty} x_{lmn} = \lim_{a \to \infty} x_{l_a m_a n_a} = x$, for all $(l,m,n) \in A$.

Let V be a neighborhood of the point x such that $U_a \subset V$.

For $(l,m,n) \in A$ with $l \ge i_a, m \ge j_a, n \ge k_a$, We can find $b \ge a$ such that

$$(i_b, j_b, k_b) \le (l,m,n) \le (i_{a+1}, j_{a+1}, k_{a+1})$$

Therefore, $(l,m,n) \in A_b$.[by the definition of A]

Then for any $(l,m,n) \in A\, with\, l \ge i_b, m \ge j_b, n \ge k_b$, $x_{lmn} \in U_b \subset U_a \subset V$

Hence, $\lim_{a \to \infty} x_{l_a m_a n_a} = x$. In other words, $x_{l_a m_a n_a} \overset{s^*}{\to} x$.

t. Theorem

Let $\{x_{lmn}\}$ be a statistically convergent triple sequence in a topological space $(X, \mathfrak{I})$. Then addition or exclusion of finitely many terms from the triple sequence doesn't affects its s-convergence.

Proof: Let $\{x_{lmn}\}$ converges statistically to $x \in X$. Then, for any open neighborhood U of x, we have $d(\{(l,m,n): x_{lmn} \in U\}) = 1$.

We now add a finite number terms, say $y_1, y_2, \ldots, y_3$ to the triple sequence $\{x_{lmn}\}$ and label the new triple sequence as $\{z_{lmn}\}$ and put the new elements as $y_k = z_{l_k m_k n_k}$

$$\text{Then, } d(\{(i,j,k): z_{ijk} \in U\}) = d\left(\left\{(i,j,k): z_{l_i m_j n_k} \in U\right\}\right) + d\left(\left\{(i,j,k): x_{ijk} \in U\right\}\right) = 0 + 1 = 1.$$

Hence, the new triple sequence after adding finite elements s-converges to the same limit.

After exclusion of finite elements from the triple sequence, this result can be established by following same technique.

u. Theorem

Let A and B be two nonempty subsets of topological space $(X, \mathfrak{I})$. Let the triple sequence $\{x_{lmn}\}$ converges statistically to $A \cup B$. Then there exists a subsequence of the triple sequence which s-converge to either A or B.

Proof: Let us consider a triple sequence $\{x_{lmn}\}$ which s-converges to x in $A \cup B$. We construct a sequence $\{z_l\}$ defined by $\{z_l = x_{lmn}\}$, for all $l \in \mathbb{N}$.

Let U be any neighborhood of x in $A \cup B$. Then, $d(\{l: z_l \in U\}) = 1$ and $\{z_l\} \in A \cup B$.

By the Theorem 3.11, at least one of the set A or B must include a non-thin subsequence of $\{z_l\}$ which statistically converges to x.

Let us suppose A is the set and $\{z_{l_k}\}$ be the subsequence.

Let us consider a triple sequence $\{y_{kmn}\}$ which is defined by $y_{kmn} = z_{l_k} \forall\, l,m,n \in \mathbb{N}$.

Once again by the Theorem 3.11, every non-thin subsequence of statistically convergent triple sequence is itself s-convergent and so $z_{l_k} \overset{s}{\to} x$

Hence $d(\{(m,n) \in \mathbb{N} \times \mathbb{N}: y_{kmn} \notin U\}) = 0$, uniformly for all k and thus $\{y_{kmn}\}$ s-converges to x.

v. Theorem:

Let $\{x_{lmn}\}$ be a triple sequence which converges statistically. If $\{x_p\}$ is a sequence constructed from $\{x_{lmn}\}$ im such a way that $x_p = x_{nml}$, for $l,m,n > p$. Then the triple sequence $\{y_{lmn}\}$ defined by $y_{lmn} = x_{l,}\, for\ all\ m,n \in \mathbb{N}$, is statistically convergent to some limit.

Proof: Since $\{x_{lmn}\}$ statistically converges to x, then for any open set U containing x we have,

$$d(\{(l,m,n,): x_{lmn} \in U\}) = 1.$$

So, $d(\{p: x_p \in U\}) = 1$, since $x_p = x_{lmn}$ for some $l,m,n > p$.

Now, $d(\{(l,m,n): y_{lmn} \notin U\}\}) = d(\mathbb{N} \times \mathbb{N} \times \mathbb{N} - \{p: x_p \in U\}) = d(\{p: x_p \in U\})=1 - 1 = 0$

Therefore, $\{y_{lmn}\}$ is also statistically convergent to x.

w. Formation of topology:

Let X be a nonempty set and $\mathfrak{F}$ be the collection of all s-convergent triple sequence whose range is X. The limit to which the triple sequence is converging statistically, let's call them the s-limit of the triple sequence. Let us define any subset $A \subseteq X$ as open set iff it satisfies the following condition:

There doesn't exist any statistically convergent triple sequence outside A whose s-limit lies inside A.

The family $\mathfrak{F}$ of such sets (what we call open sets) forms a topology on X.

Proof the above claim:

ϕ, X are obviously member of $\mathfrak{F}$, since there doesn't exists any s-convergent triple sequence outside X and ϕ can't possess any triple sequence.

Let, $\{A_\alpha: \alpha \in \Lambda, \Lambda \text{ being the index set}\}$ be the class of arbitrary numbers of open sets and

$$A = \cup_{\alpha\in\Lambda} A_\alpha.$$

Let $\{x_{lmn}\} \in X - A_\alpha, \forall\, \alpha \in \Lambda$ and since $A_\alpha \in \mathfrak{F}$, so the limit of $\{x_{lmn}\}$ doesn't belong to any of A_α and thus not in A. Therefore, A also belongs to $\mathfrak{F}$.

Let, $A_i, (i = 1,2, \dots, n)$ be finite number of open sets in $\mathfrak{F}$ and let $\{x_{lmn}\}$ be s-convergent triple sequence which lies outside $\cap_{i=1}^{n} A_i$. If possible, let this triple sequence has its s-limit with $\cap_{i=1}^{n} A_i$.

From the Theorem 3.22, at least one of $X - \cup_{i=1}^{n} A_i$ and $\left(A_p \cap \left(X - \cup_{\substack{i=1\\ i\neq p}}^{n} A_i\right)\right) \cup \left(\left(X - A_p\right) \cap \left(\cup_{\substack{i=1\\ i\neq p}}^{n} A_i\right)\right)$ must possess s-convergent triple sequence whose range is thinner that of $\{x_{lmn}\}$ and also possess the s-limit.

If $\exists\, \{y_{lmn}\} \in X - \cup_{i=1}^{n} A_i$ whose range is a subset of the range of $\{x_{lmn}\}$ having s-limit x in $\cap_{i=1}^{n} A_i \subset \cup_{i=1}^{n} A_i$, then we arrive at a contradiction that $\cup_{i=1}^{n} A_i$ is open.

So, $\left(A_p \cap \left(X - \cup_{\substack{i=1\\ i\neq p}}^{n} A_i\right)\right) \cup \left(\left(X - A_p\right) \cap \left(\cup_{\substack{i=1\\ i\neq p}}^{n} A_i\right)\right)$ contains a s-convergent triple sequence $\{y_{lmn}\}$ whose range and s-limit limit follow the same as above.

Thus at least one of $\left(A_p \cap \left(X - \bigcup_{\substack{i=1 \\ i\neq p}}^{n} A_i\right)\right)$ and $\left((X - A_p) \cap \left(\bigcup_{\substack{i=1 \\ i\neq p}}^{n} A_i\right)\right)$ contains s-convergent triple sequence say $\{z_{lmn}\}$ which follows the above condition for range compared to $\{y_{lmn}\}$.

Let $\left(A_p \cap \left(X - \bigcup_{\substack{i=1 \\ i\neq p}}^{n} A_i\right)\right)$ contains $\{z_{lmn}\}$.

Then, elements of $\{z_{lmn}\}$ in $\left(X - \bigcup_{\substack{i=1 \\ i\neq p}}^{n} A_i\right)$ having s-limit x is in $\bigcup_{\substack{i=1 \\ i\neq p}}^{n} A_i$, which is a contradiction that finite union open sets are open.

Hence, there doesn't exist any statistically convergent triple sequence in $X - \bigcup_{i=1}^{n} A_i$ which has s-limit in $\bigcap_{i=1}^{n} A_i$ and so $\bigcap_{i=1}^{n} A_i$ is open.

Therefore, $(X, \mathfrak{F})$ is a topological space.

An extensive study can be made of statistically convergent triple sequence in this topological space.

IV. Conclusion

The exploration of statistical convergence in triple sequences within topological spaces has undergone significant advancements in recent years. This study contributes to the ongoing discourse by introducing novel concepts and methodologies that enhance our understanding of convergence behaviors in complex mathematical structures. Recent literature underscores the importance of extending statistical convergence to triple sequences. For instance, Granados, Das, and Osu introduced the Mλm,n,p-statistical convergence method, providing a new framework for analyzing triple sequences in topological spaces[36]. This approach has been instrumental in establishing inclusion relations between traditional statistical convergence and the newly proposed method. Further developments have been observed in the context of non-Archimedean fields. The work by Sahiner and Tripathy delves into statistical summability for triple sequences over such fields, offering characterizations that bridge the gap between statistical convergence and summability methods[37]. This research highlights the versatility of statistical convergence concepts across different mathematical domains.

The concept of ideal convergence has also been extended to triple sequences. In their 2023 study, Sahiner et al. examined ideal convergence in random 2-normed spaces, providing insights into the behavior of triple sequences under this convergence criterion. Their findings have implications for the broader understanding of convergence in probabilistic and fuzzy environments. Advancements have not been limited to theoretical constructs. Practical applications have emerged, particularly in the analysis of sequences of functions within neutrosophic normed spaces. Khan, Khan, and investigated statistical convergence in such spaces, introducing notions of statistical pointwise and uniform convergence[38]. Their work demonstrates the applicability of statistical convergence concepts in handling uncertainty and indeterminacy in mathematical models. The study of rough convergence has also seen notable progress. Esi et al. explored the rough convergence of ρ-Cauchy sequences of triple sequences, examining the relationships between cluster points and rough limit points[39]. This research provides a deeper understanding of convergence behaviors in sequences that do not conform to traditional convergence criteria.

Moreover, the integration of statistical convergence with Wijsman convergence has been a focal point of recent studies. Esi and Subramanian introduced the concept of Wijsman rough statistical convergence for triple sequences, establishing criteria for convergence and examining the properties of the set of limit points[40]. Their findings contribute to the broader discourse on convergence in metric and topological spaces. The intersection of statistical convergence with fuzzy and probabilistic frameworks has also been explored. Altaweel et al. adapted ideal convergence theories to fuzzy metric spaces, expanding the applicability of statistical convergence concepts to environments characterized by uncertainty and imprecisión[41]. In the realm of topology, the work of Zhou, Liu, and Lin on topological spaces defined by convergence offers a fresh perspective on how convergence concepts can define and influence topological structures[42]. Their research underscores the foundational role of convergence in shaping the properties and behaviors of topological spaces.

The cumulative insights from these studies underscore the dynamic and evolving nature of statistical convergence in triple sequences. The integration of new methodologies, the extension to various mathematical frameworks, and the exploration of practical applications collectively enhance our comprehension of convergence phenomena. As research continues to advance, it is anticipated that these developments will foster further innovations in mathematical analysis and its applications across diverse scientific fields.

References

[1] H. Fast, «Sur la convergence statistique», *Colloquium Mathematicum*, vol. 2, pp. 241-244, 1951, doi: 10.4064/cm-2-3-4-241-244.

[2] I. J. Schoenberg, «The Integrability of Certain Functions and Related Summability Methods», *The American Mathematical Monthly*, vol. 66, no. 5, pp. 361-375, 1959, doi: 10.2307/2308747.

[3] R. C. Buck, «Generalized Asymptotic Density», *American Journal of Mathematics*, vol. 75, no. 2, pp. 335-346, 1953, doi: 10.2307/2372456.

[4] J. A. Fridy, «On Statistical Convergence», *Analysis*, vol. 5, no. 4, pp. 301-313, 1985, doi: 10.1524/anly.1985.5.4.301.

[5] T. Šalát, «On statistically convergent sequences of real numbers», *Mathematica Slovaca*, vol. 30, no. 2, pp. 139-150, 1980.

[6] S. A. Mohiuddine y M. A. Alghamdi, «Statistical summability through a lacunary sequence in locally solid Riesz spaces», *Journal of Inequalities and Applications*, vol. 2012, art. no. 225, 2012, doi: 10.1186/1029-242X-2012-225.

[7] H. Albayrak y S. Pehlivan, «Statistical convergence and statistical continuity on locally solid Riesz spaces», *Topology and its Applications*, vol. 159, núm. 7, pp. 1887-1893, 2012, doi: 10.1016/j.topol.2012.03.014.

[8] H. Çakalli, «On Statistical Convergence in Topological Groups», *Pure and Applied Mathematika Sciences*, vol. 43, no. 1-2, pp. 27-31, 1996.

[9] G. Di Maio y L. D. R. Kočinac, «Statistical convergence in topology», *Topology and its Applications*, vol. 156, núm. 1, pp. 28-45, 2008, doi: 10.1016/j.topol.2008.01.015.

[10] S. Karakuş, K. Demirci y O. Duman, «Statistical convergence on intuitionistic fuzzy normed spaces», *Chaos, Solitons & Fractals*, vol. 35, núm. 4, pp. 763-769, 2008, doi: 10.1016/j.chaos.2006.05.046.

[11] D. Kumar y M. Mursaleen, «λ-statistical convergence in intuitionistic fuzzy normed spaces», *Filomat*, vol. 25, núm. 4, pp. 109-119, 2011, doi: 10.2298/FIL1104109K.

[12] S. A. Mohiuddine y E. Savaş, «Lacunary statistically convergent double sequences in probabilistic normed spaces», *Annali dell'Università di Ferrara*, vol. 58, núm. 2, pp. 331-339, 2012, doi: 10.1007/s11565-012-0157-5.

[13] I. J. Maddox, «Statistical convergence in a locally convex space», *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 104, no. 1, pp. 141-145, 1988, doi: 10.1017/S0305004100065312.

[14] B. C. Tripathy, «Statistically convergent double sequences», *Tamkang Journal of Mathematics*, vol. 34, no. 3, pp. 231-237, 2003, doi: 10.5556/j.tkjm.34.2003.314

[15] M. Mursaleen y O. H. H. Edely, «Statistical convergence of double sequences», *Journal of Mathematical Analysis and Applications*, vol. 288, núm. 1, pp. 223-231, 2003, doi: 10.1016/S0022-247X(03)00580-0.

[16] F. Móricz, «Statistical convergence of multiple sequences», *Archiv der Mathematik*, vol. 81, no. 1, pp. 82-89, 2003, doi: 10.1007/s00013-003-0506-9.

[17] Belen, M. Mursaleen y M. Yildirim, «Statistical A-summability of double sequences and a Korovkin type approximation theorem», *Bulletin of the Korean Mathematical Society*, vol. 49, núm. 4, pp. 851-861, 2012, doi: 10.4134/BKMS.2012.49.4.851.

[18] K. Demirci y S. Karakuş, «Statistical A-summability of positive linear operators», *Mathematical and*

Computer Modelling, vol. 49, núm. 1-2, pp. 189-195, 2009, doi: 10.1016/j.mcm.2008.05.053.

[19] K. Demirci y S. Karakuş, «Korovkin-Type Approximation Theorem for Double Sequences of Positive Linear Operators via Statistical A-Summability», *Results in Mathematics*, vol. 63, pp. 1-13, 2013, doi: 10.1007/s00025-011-0140-y

[20] O. H. H. Edely y M. Mursaleen, «On Statistical A-Summability», *Mathematical and Computer Modelling*, vol. 49, núm. 3-4, pp. 672-680, 2009, doi: 10.1016/j.mcm.2008.05.053.

[21] M. Mursaleen, C. Çakan, S. A. Mohiuddine y E. Savaş, «Generalized statistical convergence and statistical core of double sequences», *Acta Mathematica Sinica, English Series*, vol. 26, núm. 11, pp. 2131-2144, 2010, doi: 10.1007/s10114-010-9050-2.

[22] S. Karakuş y K. Demirci, «Statistical convergence of double sequences on probabilistic normed spaces», *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Art. ID 14737, 2007, doi: 10.1155/2007/14737.

[23] S. A. Mohiuddine y E. Savaş, «Lacunary statistical convergence in probabilistic normed spaces», *Mathematical and Computer Modelling*, vol. 55, núm. 1-2, pp. 164-173, 2012, doi: 10.1016/j.mcm.2011.02.008.

[24] E. Savaş y S. A. Mohiuddine, «λ-statistically convergent double sequences in probabilistic normed spaces», *Mathematica Slovaca*, vol. 62, núm. 1, pp. 99-108, 2012, doi: 10.2478/s12175-011-0075-5.

[25] B. C. Tripathy y P. K. Nath, «Statistical convergence of complex uncertain sequences», *New Mathematics and Natural Computation*, vol. 13, núm. 3, pp. 359-374, 2017, doi: 10.1142/S1793005717500090.

[26] B. Das, B. C. Tripathy, P. Debnath y B. Bhattacharya, «Statistical convergence of complex uncertain triple sequence», *Communications in Statistics - Theory and Methods*, vol. 51, núm. 20, pp. 7088-7100, 2022, doi: 10.1080/03610926.2020.1871016.

[27] B. Das, B. C. Tripathy, P. Debnath y B. Bhattacharya, «Characterization of statistical convergence of complex uncertain double sequence», *Analysis and Mathematical Physics*, vol. 10, núm. 4, pp. 1-20, 2020, doi: 10.1007/s13324-020-00419-7.

[28] B. Das, P. Debnath y B. C. Tripathy, «On statistically convergent complex uncertain sequences», *Carpathian Mathematical Publications*, vol. 14, núm. 1, pp. 135-146, 2022, doi: 10.15330/cmp.14.1.135-146

[29] B. Das, B. C. Tripathy y P. Debnath, «A new type of statistically convergent complex uncertain triple sequence», *Thai Journal of Mathematics*, vol. 20, núm. 3, pp. 1441-1450, sep. 2022.

[30] B. Das y B. C. Tripathy, «On λ^2-statistical convergence of complex uncertain sequences», *Asian-European Journal of Mathematics*, vol. 16, núm. 5, Art. no. 2350083, 2023, doi: 10.1142/S179355712 3500833.

[31] S. Karakuş, K. Demirci y O. Duman, «Statistical convergence on intuitionistic fuzzy normed spaces», *Chaos, Solitons & Fractals*, vol. 35, núm. 4, pp. 763-769, 2008, doi: 10.1016/j.chaos.2006. 05.046.

[32] B. Das, P. Debnath y B. C. Tripathy, «On statistically convergent complex uncertain sequences», *Carpathian Mathematical Publications*, vol. 14, núm. 1, pp. 135-146, 2022, doi: 10.15330/cmp.14.1.135-146.

[33] B. C. Tripathy y P. K. Nath, «Statistical convergence of sequences of uncertain variables», *AIP Conference Proceedings*, vol. 1836, Art. no. 020068, 2017, doi: 10.1063/1.4981963.

[34] V. Kumar y M. Mursaleen, «On (λ, μ)-statistical convergence of double sequences on intuitionistic fuzzy normed spaces», *Filomat*, vol. 25, núm. 2, pp. 109-120, 2011, doi: 10.2298/FIL1102109K.

[35] A. Sahiner, M. Gurdal y F. K. Duden, «Triple sequences and their statistical convergence», *Selçuk Journal of Applied Mathematics*, vol. 8, núm. 2, pp. 49-55, 2007.

[36] C. Granados, A. K. Das y B. O. Osu, «Mλ, -statistical convergence for triple sequences», *Journal of Analysis*, vol. 30, pp. 451-468, 2022, doi: 10.1007/s41478-021-00355-0.

[37] A. Sahiner y B. C. Tripathy, «Statistical and ideal convergences in topology», *Mathematics*, vol. 11, núm. 3, art. 663, 2023, doi: 10.3390/math11030663.

[38] V. A. Khan, M. D. Khan y A. Kumar, «Statistical convergence of sequences of functions in neutrosophic normed spaces», *Numerical Functional Analysis and Optimization*, vol. 44, núm. 13, pp. 1443-1463, 2023, doi: 10.1080/01630563.2023.2248695.

[39] A. Esi, M. Aiyub y N. Subramanian, «On rough convergence of ρ-Cauchy sequence of triple sequences», *Analysis*, vol. 43, núm. 4, pp. 321-335, 2023, doi: 10.1515/anly-2018-0075.

[40] A. Esi y N. Subramanian, «Wijsman rough statistical convergence on triple sequences», *International Journal of Sciences: Basic and Applied Research (IJSBAR)*, vol. 32, núm. 3, pp. 13-27, 2017.

[41] A. Altaweel, M. Rashid y L. D. R. Kočinac, «Ideal convergence in fuzzy metric spaces», *Mathematics*, vol. 11, núm. 3, art. 663, 2023, doi: 10.3390/math11030663.

[42] X. Zhou, L. Liu y S. Lin, «On topological spaces defined by $\mathcal{I}$-convergence», *Bulletin of the Iranian Mathematical Society*, vol. 46, núm. 3, pp. 675-692, 2020, doi: 10.1007/s41980-020-00445-1.